

# From Stäckel systems to integrable hierarchies of PDE's: Benenti class of separation relations

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## Abstract

We propose a general scheme of constructing of soliton hierarchies from finite dimensional Stäckel systems and related separation relations. In particular, we concentrate on the simplest class of separation relations, called Benenti class, i.e. certain Stäckel systems with quadratic in momenta integrals of motion.

## 1 Introduction

The theory of integrable nonlinear evolution equations has a long history as a part of many branches of theoretical physics and applied mathematics. Generally it can be divided in two parts: the theory of integrable nonlinear ordinary differential equations (ODE's) and the theory of integrable nonlinear partial differential equations (PDE's). Within the first class of equations (ODE's) we will consider finite dimensional Hamiltonian systems, integrable by the Hamilton-Jacobi method, called *Stäckel systems*, while within the second class (PDE's) we will consider (1+1)-dimensional field systems, having infinite hierarchy of commuting symmetries and called further for simplicity *soliton systems*. The solvability by quadratures of some class of finite dimensional systems by the Hamilton-Jacobi method, laid in the 19-th century one of the fundamentals of analytical mechanics of integrable systems, while the solvability by quadratures of some class of infinite dimensional field systems by the Inverse Scattering Method, laid in second half of the 20-th century one of the fundamentals of the so called soliton theory.

During the last few decades both theories have been developed very intensively using many common modern mathematical tools like Lax representation, r-matrix theory, multi-Hamiltonian theory etc. In that time some links between both theories were investigated. It was found ([1]-[4], see also references in [5]) that finite dimensional restrictions, invariant with respect to the action of a given soliton system, like stationary flows, restricted flows or constrained flows of Lax representation, are Liouville integrable Hamiltonian systems of Stäckel type. Moreover, analytical solutions of an appropriate finite dimensional systems are closely related to a special class of solutions of related soliton systems, like for example so-called finite-gap solutions [6],[7].

In the present paper we are interested in passing in the opposite direction - building integrable hierarchies of PDE's from Stäckel systems [8]. In that sense we would like to initiate a unified approach to Stäckel ODE's and soliton PDE's. Our claim is the following: *both a wide class of Stäckel systems and*

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a wide class of soliton systems can be constructed from common fundamental objects known as separation relations (or from separation curves).

The paper is organized as follows. Section 2 is devoted to general description of the concept of separation relations. Section 3 explains the main ideas of relating soliton systems with separation curves that are quadratic in momenta. The idea is to apply to a set of Killing vector fields a set of invariants generated by Euler-Lagrange equations associated with appropriately chosen Lagrangian densities. This allows for elimination of some variables in our Killing systems which leads to dispersive soliton hierarchies. Section 4 is a brief introduction to what can be called Benenti class of Stäckel systems. In section 5 we describe the structure of our systems in Viète coordinates. In Section 6 we explain the details of our elimination procedure which allows, in a systematic way, to construct soliton hierarchies. It is divided into two subsections as the elimination procedure differs in case of "positive" and "negative" (see below) separable potentials. Section 7 concludes the article with several examples.

## 2 Separation relations

Let us consider a  $2n$ -dimensional manifold  $M$  equipped with a Poisson operator  $\Pi$  with some canonical (Darboux) coordinates labelled as  $M \ni u = (\mu, \lambda)$ , with  $\mu = (\mu_1, \dots, \mu_n)$  and  $\lambda = (\lambda_1, \dots, \lambda_n)$ . The following definition introduces the basic object of our considerations [9]:

**Definition 1** A set of  $n$  relations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \dots, a_n) = 0, \quad i = 1, \dots, n, \quad a_i \in \mathbf{R} \quad (1)$$

(each involving only one pair  $\lambda_i, \mu_i$  of canonical coordinates) are called separation relations provided that the dependence of  $\varphi$  on  $a$  is essential i.e. that  $\det \left( \frac{\partial \varphi_i}{\partial a_j} \right) \neq 0$ .

The condition in (1) means that we can resolve the equations (1) with respect to  $a_i$  obtaining  $a_i = H_i(\lambda, \mu)$ ,  $i = 1, \dots, n$ . This defines some new functions  $H_i(\lambda, \mu)$  that in turn define the following Hamiltonian systems (evolutionary vector fields) on  $M$

$$u_{t_i} = \Pi dH_i = X_{H_i}, \quad i = 1, \dots, n. \quad (2)$$

If the functions  $W_i(\lambda_i, a)$  are solutions of a system of  $n$  decouple ODE's obtained from (1) by substituting  $\mu_i = \frac{\partial W_i(\lambda_i, a)}{\partial \lambda_i}$

$$\varphi_i \left( \lambda_i, \mu_i = \frac{\partial W_i(\lambda_i, a)}{\partial \lambda_i}, a_1, \dots, a_n \right) = 0, \quad i = 1, \dots, n, \quad (3)$$

then the function  $W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a)$  is an additively separable solution of *all* the equations (3) and *simultaneously* it is a solution of all Hamilton-Jacobi equations

$$a_i = H_i \left( \lambda, \mu = \frac{\partial W(\lambda, a)}{\partial \lambda} \right), \quad i = 1, \dots, n \quad (4)$$

related with the Hamiltonians  $H_i$  - simply because solving (1) to the form  $a_i = H_i(\lambda, \mu)$  is a purely algebraic operation. Assume now that  $\det \left( \frac{\partial^2 W}{\partial \lambda_i \partial a_j} \right) = \det \left( \frac{\partial^2 W_i}{\partial \lambda_i \partial a_j} \right) \neq 0$ . Then the Hamiltonians  $H_i$  Poisson-commute since the constructed function  $W(\lambda, a)$  is a generating function for the canonical transformation  $(\lambda, \mu) \rightarrow (b, a)$  where

$$b_i = \frac{\partial W(\lambda, a)}{\partial a_i} = t_i + \text{const}_i, \quad i = 1, \dots, n. \quad (5)$$

Equations (5) are implicit solutions of (2) known as the *inversion Jacobi problem*. Thus, starting from a set of  $n$  separation relations we can create an  $n$ -dimensional separable Liouville system. All systems separable in the sense of Hamilton-Jacobi theory can be obtained in this way.

In an important case, when the functions  $\varphi_i$  in (1) do not depend on the index  $i$ , the separation relations (1) can be generated by taking  $n$  copies of a curve in  $\lambda$ - $\mu$  plane:

$$\varphi(\lambda, \mu, a_1, \dots, a_n) = 0, \quad a_i \in \mathbf{R} \quad (6)$$

called *separation curve*.

Restricting our considerations to a subclass of (1), when all separation relations are affine in  $a_i = H_i$  with coefficients being monomials in  $\lambda$  and  $\mu$ , we obtain

$$\sum_{k=1}^n H_k \mu_i^{\alpha_k} \lambda_i^{\beta_k} = \psi_i(\lambda_i, \mu_i), \quad i = 1, \dots, n, \quad \alpha_k, \beta_k \in \mathbf{N} \quad (7)$$

where  $\psi_i$  are arbitrary smooth functions of two arguments. Equations (7) are called generalized *Stäckel separation relations* and the related dynamic systems, generated by Hamiltonian functions  $H_i$ , are called the *Stäckel separable* ones. To recover explicit Stäckel form of Hamiltonians it is sufficient to solve the linear system (7) with respect to  $H_i$ . If additionally  $\psi_i(\lambda_i, \mu_i) = \psi(\lambda_i, \mu_i)$  then the above separation conditions can be represented by  $n$  copies of the following separation curve:

$$\sum_{k=1}^n H_k \mu^{\alpha_k} \lambda^{\beta_k} = \psi(\lambda, \mu). \quad (8)$$

The separable systems that were most intensively studied in the last century were one-particle dynamical systems on Riemannian manifolds with flat or constant curvature metrics. All these systems can be obtained by choosing  $\alpha_i = 0, \beta_i = n - i, i = 1, \dots, n$  with  $\psi$  quadratic in momenta

$$\psi(\lambda, \mu) = \frac{1}{2} f(\lambda) \mu^2 + \gamma(\lambda). \quad (9)$$

This case will be considered in the next sections of this article.

We can now shortly present - by a simple example - the possibility of passing from a separation curve to soliton systems [10]. Let us consider the separation curve (8) with  $n = 2, \alpha_1 = \alpha_2 = 0, \beta_1 = 1, \beta_2 = 0$  and with  $\psi$  in the form of (9)

$$H_1 \lambda + H_2 = \frac{1}{2} \lambda \mu^2 + \lambda^4. \quad (10)$$

The related separation conditions (7) are

$$\begin{cases} H_1 \lambda_1 + H_2 = \frac{1}{2} \lambda_1 \mu_1^2 + \lambda_1^4 \\ H_1 \lambda_2 + H_2 = \frac{1}{2} \lambda_2 \mu_2^2 + \lambda_2^4 \end{cases} \quad (11)$$

Solving this linear system with respect to  $H_1$  and  $H_2$  one gets the Liouville integrable system (2) on four dimensional phase space, written in separation coordinates  $(\lambda, \mu)$ . The explicit form of Hamiltonians  $H_i$  is

$$H_1 = \frac{1}{2} \frac{\lambda_1 \mu_1^2 - \lambda_2 \mu_2^2 + 2\lambda_1^4 - 2\lambda_2^4}{\lambda_1 - \lambda_2}, \quad H_2 = \frac{1}{2} \frac{\lambda_1 \lambda_2 (\mu_1^2 - \mu_2^2 + 2\lambda_1^3 - 2\lambda_2^3)}{\lambda_2 - \lambda_1}$$

The canonical transformation of the form

$$q_1 = \lambda_1 + \lambda_2, \quad \frac{1}{4} q_2^2 = -\lambda_1 \lambda_2,$$

$$p_1 = \frac{\lambda_1 \mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2 \mu_2}{\lambda_2 - \lambda_1}, \quad p_2 = \sqrt{-\lambda_1 \lambda_2} \left( \frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1} \right)$$

transforms the system to new coordinates  $(q, p)$ , with

$$H_1 = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + q_1^3 + \frac{1}{2} q_1 q_2^2, \quad H_2 = \frac{1}{2} q_2 p_1 p_2 - \frac{1}{2} q_1 p_2^2 + \frac{1}{16} q_2^4 + \frac{1}{4} q_1^2 q_2^2.$$

The function  $H_1(q, p)$  turns out to be the Hamiltonian function of the integrable case of the Henon-Heiles system, while  $H_2(q, p)$  is the additional involutive first integral of this system. Let us now denote the evolution parameters  $t_1$  and  $t_2$  of the system by  $x$  and  $t$ , respectively. Then we obtain

$$\begin{aligned} q_{1,x} &= \frac{\partial H_1}{\partial p_1} = p_1, & q_{2,x} &= \frac{\partial H_1}{\partial p_2} = p_2, \\ q_{1,t} &= \frac{\partial H_2}{\partial p_1} = \frac{1}{2}q_2 p_2, & q_{2,t} &= \frac{\partial H_2}{\partial p_2} = \frac{1}{2}q_2 p_1 - q_1 p_2, \end{aligned}$$

from which eliminating  $p_1$  and  $p_2$  we obtain a system of first order PDE's for  $q_1(x, t)$  and  $q_2(x, t)$

$$q_{1,t} = \frac{1}{2}q_2 q_{2,x} = \frac{1}{4}(q_2^2)_x, \quad q_{2,t} = \frac{1}{2}q_2 q_{1,x} - q_1 q_{2,x}. \quad (12)$$

Finally, we can eliminate  $q_2$  through

$$q_{1,xx} = p_{1,x} = -\frac{\partial H_1}{\partial q_1} = -3q_1^2 - \frac{1}{2}q_2^2$$

which yields  $q_2^2 = -6q_1^2 - 2q_{1,xx}$  and then generate a higher order (in  $x$ -derivatives) PDE. The first equation in (12) turns then into the famous KdV soliton system

$$q_{1,t} + \frac{1}{2}q_{1,xxx} + 3q_1 q_{1,x} = 0, \quad (13)$$

while the second equation in (12) turns into a differential consequence of the first one. Obviously, just from the presented construction, there is no guarantee that equation (13) is integrable. We can only say that  $q_1(x, t)$  calculated from the corresponding inverse Jacobi problem is a nontrivial particular solution (one-gap solution) for the field system (13). To prove the integrability of (13) one has to construct some related infinite hierarchy of symmetries using some more regular procedure.

### 3 From separation curves to constrained dispersionless systems

In this paper we will concentrate on a special but important class of separation curves with the function  $\psi(\lambda, \mu)$  in (8) being quadratic in momenta  $\mu$ , (more precisely, of the form (9)) and with multipliers of Hamiltonian functions being monomials with respect to  $\lambda$

$$H_1 \lambda^{\beta_1} + \dots + H_n \lambda^{\beta_n} = \frac{1}{2} \lambda^m \mu^2 + \lambda^k, \quad (14)$$

where  $\beta_1 > \dots > \beta_{n-1} > \beta_n = 0$ ,  $\beta_i \in \mathbf{N}$ ,  $m, k \in \mathbf{Z}$  and  $n \in \mathbf{N}$ . Separable systems from this class describe one-particle dynamics on Riemannian manifolds and belong to classical Stäckel systems. Each class of these systems is labelled by a decreasing sequence  $(\beta_1, \dots, \beta_n)$  while members of a given class are numbered by pairs  $(m, k) \in \mathbf{Z}^2$ . Taking  $n$  copies of the curve (14) with variables  $(\lambda, \mu)$  labelled within each copy as  $(\lambda_i, \mu_i)$ , we obtain a system of  $n$  separation relations in the form of  $n$  equations linear in the coefficients  $H_i$ . Solving it we obtain  $n$  functions  $H_r^{(m,k)} = H_r^{(m,k)}(\lambda, \mu)$  of the form

$$H_r^{(m,k)} = \frac{1}{2} \mu^T K_r G^{(m)} \mu + V_r^{(k)}, \quad r = 1, \dots, n, \quad m, k \in \mathbf{Z} \quad (15)$$

where we denote  $\lambda = (\lambda_1, \dots, \lambda_n)^T$  and  $\mu = (\mu_1, \dots, \mu_n)^T$ . The functions (15) can be interpreted as  $n$  Hamiltonians on the phase space  $T^*\mathcal{Q}$  cotangent to a Riemannian manifold  $\mathcal{Q}$  equipped with the contravariant metric tensor  $G^{(m)}$ . These Hamiltonians are in involution with respect to the canonical Poisson bracket on  $T^*\mathcal{Q}$ . Moreover, they are separable in the sense of Hamilton-Jacobi theory since they by the very definition satisfy Stäckel relations (14). The objects  $K_r$  in (15) can be interpreted as (1,1)-type Killing tensors on  $\mathcal{Q}$ . The scalar functions  $V_r^{(k)}$  are separable potentials. Further, all the metric tensors  $G^{(m)}$  and all the Killing tensors  $K_r$  are diagonal in  $\lambda$ -variables so that:

$$K_r = \text{diag}(v_r^1, \dots, v_r^n) \quad (16)$$

where  $v_r^i$  are eigenvalues of  $K_r$ . We will constantly assume that these eigenvalues are single.

The set (15) of  $n$  Hamiltonian functions leads to  $n$  Hamiltonian systems on  $T^*\mathcal{Q}$  of the form

$$\lambda_{t_r} = \frac{\partial H_r^{(k,m)}}{\partial \mu}, \quad \mu_{t_r} = -\frac{\partial H_r^{(k,m)}}{\partial \lambda}, \quad r = 1, \dots, n. \quad (17)$$

Let us now call the variable  $t_1$  as  $x$ ;  $t_1 \equiv x$ . Since all the Hamiltonians  $H_r^{(k,m)}$  (for fixed  $k$  and  $m$ ) commute, the equations (17) have a common set of solutions depending on all the evolution parameters  $t_i$

$$\lambda_i = \lambda_i(t_1 = x, t_2, \dots, t_n), \quad \mu_i = \mu_i(t_1 = x, t_2, \dots, t_n).$$

We have, due to (17), that

$$\lambda_x \equiv \lambda_{t_1} = \frac{\partial H_1^{(k,m)}}{\partial \mu} = G^{(m)} \mu \quad \text{so that} \quad \mu = g^{(m)} \lambda_x,$$

where the inverse of  $G^{(m)}$  (i.e. respective covariant metric tensors) is denoted as  $g^{(m)}$ . Observe that the above relation does not depend on  $k$ . Using this to eliminate  $\mu$  from the first part of (17) we obtain

$$\lambda_{t_r} = \frac{\partial H_r^{(k,m)}}{\partial \mu} = K_r G^{(m)} \mu,$$

or, according to the above

$$\lambda_{t_r} = K_r \lambda_x \equiv Z_r^n(\lambda, \lambda_x), \quad r = 1, \dots, n. \quad (18)$$

This is a set of  $n$  autonomous systems of  $n$  coupled first order PDE's of evolutionary type, with the right hand sides depending linearly on the derivatives  $\lambda_x$  [11]. More precisely, it is a set of  $n$  integrable dispersionless equations, belonging to the class of so-called weakly nonlinear semi-Hamiltonian systems [12],[13], where the variables  $\lambda_i$  are the Riemann invariants for (18). We will call them *Killing dispersionless system* as they are constructed directly from Killing tensors.

We will interpret the right hand sides of (18) as vector fields on an infinite dimensional manifold  $\mathcal{M}$  the points of which are vector functions of  $x$  of the form  $u = (\lambda_1(x), \dots, \lambda_n(x))$ , where we assume that the functions  $\lambda_i(x)$  are either periodic in  $x$  or they vanish together with all their derivatives when  $x \rightarrow \pm\infty$ . A vector field  $X$  is at a point  $u \in \mathcal{M}$  given by an  $n$ -tuple of the form  $X(u) = (f_1[\lambda], \dots, f_n[\lambda])$  where  $f_i[\lambda] = f_i(\lambda_1, \lambda_{1,x}, \dots, \lambda_2, \lambda_{2,x}, \dots, \lambda_n, \lambda_{n,x}, \dots)$  are differential functions of  $\lambda$ . Similarly, a covector field  $\alpha$  on  $\mathcal{M}$  is in a point  $u = (\lambda_1(x), \dots, \lambda_n(x))$  given by  $\alpha(u) = (g_1[\lambda], \dots, g_n[\lambda])$ . The dual map between  $T_u\mathcal{M}$  and  $T_u^*\mathcal{M}$  is given by

$$\langle \alpha, X \rangle(u) = \int_x \sum_{i=1}^n f_i[\lambda] g_i[\lambda] dx.$$

Here and below the integration is performed over one period (in case of periodic boundary conditions) or over  $\mathbf{R}$  in case of functions vanishing at  $\pm\infty$ . All functions and expressions are always assumed to be integrable. For any two given vector fields  $X$  and  $Y$  on  $\mathcal{M}$  their commutator is defined in a usual way as  $[X, Y] = X'[Y] - Y'[X]$  where  $X'[Y]$  denotes the directional derivative of  $X$  in the direction of  $Y$ .

As was shown in [14], the vector fields  $Z_i^n$  pairwise commute:

$$[Z_i^n, Z_j^n] = 0 \quad i, j = 1, \dots, n,$$

thus, (18) is a set of  $n$  commuting evolutionary dynamic systems (vector fields) on  $\mathcal{M}$ . We will need the superscript  $n$  to indicate the number of components (dimension) of these systems. Below we will introduce invariants on (18) that eventually turn these systems into hierarchies of soliton systems with lower number of fields. This is the main idea of this paper.

We begin by defining the following differential functions (currents, 'Lagrangians'):

$$\mathcal{L}_r^{(n,m,k)} \stackrel{\text{def}}{=} \frac{1}{2} \lambda_x^T g^{(m)} K_r \lambda_x - V_r^{(k)}, \quad r = 1, \dots, n. \quad (19)$$

In our further considerations we will especially need the first current  $\mathcal{L}_1^{(n,m,k)}$ , so we will denote it simply by  $\mathcal{L}^{(n,m,k)}$ ,

$$\mathcal{L}^{(n,m,k)} \stackrel{\text{def}}{=} \mathcal{L}_1^{(n,m,k)}$$

This current is a Legendre transform of  $H_1^{(n,m,k)}$  (this is not true for  $H_r^{(n,m,k)}$  with  $r > 1$ ). These differential functions yield the following functionals on  $\mathcal{M}$ :

$$I_r^{(n,m,k)}(u) \stackrel{\text{def}}{=} \int_x \mathcal{L}_r^{(n,m,k)}[\lambda] dx,$$

where, as usual,  $u = (\lambda_1(x), \dots, \lambda_n(x))$ . We have, of course,

$$\frac{dI_r^{(n,m,k)}}{dt_s} = \left\langle \frac{\delta I_r^{(n,m,k)}}{\delta \lambda}, Z_s^n[\lambda] \right\rangle = \left\langle E\left(\mathcal{L}_r^{(n,m,k)}\right), Z_s^n[\lambda] \right\rangle$$

where  $E = (E_1, \dots, E_n) = \left(\frac{\delta}{\delta \lambda_1}, \dots, \frac{\delta}{\delta \lambda_n}\right)$  is the Euler-Lagrange operator on  $\mathcal{M}$ .

**Lemma 2** *In the notation as above,*

$$\frac{dI_1^{(n,m,k)}}{dt_r} = 0, \quad r = 1, \dots, n.$$

**Proof.** It suffices to prove that

$$\sum_{i=1}^n E_i \left( \mathcal{L}^{(n,m,k)} \right) \lambda_{i,t_r} = \sum_{i=1}^n E_i \left( \mathcal{L}_r^{(n,m,k)} \right) \lambda_{i,x} \quad (20)$$

since integrating of (20) yields,

$$\frac{dI_1^{(n,m,k)}}{dt_r} = \frac{dI_r^{(n,m,k)}}{dt_1}, \quad r = 1, \dots, n$$

while

$$\frac{dI_r^{(n,m,k)}}{dt_1} = \int_x \sum_{i=1}^n E_i \left( \mathcal{L}_r^{(n,m,k)} \right) \lambda_{i,x} dx = \int_x \frac{d}{dx} \left( \mathcal{L}_r^{(n,m,k)} \right) dx = 0.$$

due to the appropriately chosen boundary conditions. The proof of (20) can be found in Appendix A. ■

**Corollary 3** *Lemma 2, due to theorem of [15] (see also [16]) implies that the  $2n$ -dimensional set  $\mathcal{E} \subset \mathcal{M}$  defined as*

$$\mathcal{E} = \left\{ u : E_i \left( \mathcal{L}^{(n,m,k)} \right) = 0 \text{ for all } i = 1, \dots, n \right\}$$

*is  $Z_r^n$ -invariant for all  $r = 1, \dots, n$ .*

Thus, if  $u_0 \in \mathcal{E}$  then the integral (Fröbenius)  $n$ -dimensional submanifold  $\mathcal{S}_{u_0}$  of  $\mathcal{M}$  spanned by the commuting vector fields  $Z_r^n$  and containing  $u_0$  is a subset of  $\mathcal{E}$ . This means that the solution  $\lambda(x, t_r)$  of the  $r$ -th Killing system in (18) that starts at a point  $u_0 \in \mathcal{E}$ , i.e. initially satisfying the set of Euler-Lagrange equations

$$E_i \left( \mathcal{L}^{(n,m,k)} \right) = 0, \quad i = 1, \dots, n \quad (21)$$

remains in  $\mathcal{E}$ . i.e. always satisfy (21). This further means that we can use the set of equations (21) to eliminate some of the variables  $\lambda_i$  in (18). Such an operation does not alter (18), but reparametrizes it, leading to fewer equations of higher order, and the dispersion will occur. As we will see below, this operation of elimination of variables from (18) through the use of (21) will lead both to known and new soliton hierarchies in  $(1+1)$  dimensions.

## 4 Benenti class of Stäckel systems

In the rest of this paper we consider the simplest class of separation curves (14) in the form

$$H_1 \lambda^{n-1} + H_2 \lambda^{n-2} + \dots + H_n = \frac{1}{2} \lambda^m \mu^2 + \lambda^k \quad (22)$$

( $\lambda, \mu \in \mathbf{R}$  for a moment), where  $n \in \mathbf{N}$  while  $m, k \in \mathbf{Z}$ . This object contains a complete information about the so-called Benenti systems [17]-[19]. Hamiltonian functions calculated from the related system of separation relations take the form (15) [20]. Due to a special form of (22) it turns out that the metric tensors  $G^{(m)}$  are now

$$G^{(m)} = L^m G^{(0)}, \text{ with } G^{(0)} = \text{diag} \left( \frac{1}{\Delta_1}, \dots, \frac{1}{\Delta_n} \right), \quad m \in \mathbf{Z},$$

where  $\Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j)$  and where  $L = \text{diag}(\lambda_1, \dots, \lambda_n)$  is a  $(1, 1)$ -tensor on  $\mathcal{Q}$  (it is a conformal Killing tensor with respect to all the metrics  $G^{(m)}$ ). Moreover, Killing tensors  $K_r$  can now be obtained by the following recursion relation:

$$K_{r+1} = LK_r + q_r I, \quad K_1 = I, \quad K_{n+1} = 0, \quad r = 1, \dots, n, \quad (23)$$

so that indeed they are diagonal (in  $\lambda$ -coordinates) in accordance with (16):  $K_r = \text{diag}(v_r^1, \dots, v_r^n)$ . The functions  $q_r = q_r(\lambda)$  are coefficients of the characteristic polynomial of the tensor  $L$  i.e. they are defined by

$$\det(\lambda I - L) = \sum_{i=0}^n q_i \lambda^{n-i}, \quad (24)$$

so that  $q_0 = 1, q_1 = -\sum_{i=1}^n \lambda_i, \dots, q_n = (-1)^n \prod_{i=1}^n \lambda_i$  ( $q_i$  are Viète polynomials in the variables  $\lambda$ ). Moreover, the potentials  $V_r^{(k)}$  in Hamiltonians (15) can now be obtained from the following recursion relation [20]:

$$V_r^{(k)} = V_{r+1}^{(k-1)} - q_r V_1^{(k-1)}, \quad k \in \mathbf{Z} \quad (25)$$

(with the convention that  $V_r^{(k)} \equiv 0$  for  $r < 1$  or  $r > n$ ) with the initial condition:

$$V_r^{(0)} = \delta_{r,n}, \quad r = 1, \dots, n. \quad (26)$$

This recursion can be reversed. The inverse recursion is given by

$$V_r^{(k)} = V_{r-1}^{(k+1)} - \frac{q_{r-1}}{q_n} V_n^{(k+1)}, \quad k \in \mathbf{Z}, \quad r = 1, \dots, n. \quad (27)$$

The first potentials are rather trivial:

$$V_r^{(k)} = \delta_{r,n-k} \quad \text{for } k = 0, 1, \dots, n-1, \quad V_r^{(n)} = -q_r, \quad V_r^{(-1)} = -\frac{q_{r-1}}{q_n}, \quad (28)$$

but for  $r < -1$  or for  $r > n$  the potentials become complicated polynomial (for  $r > n$ ) or rational (for  $r < -1$ ) functions of  $q$ .

From (25) we get

$$V_n^{(k)} = -q_n V_1^{(k-1)}, \quad k \in \mathbf{Z} \quad (29)$$

and

$$V_r^{(k)} = -q_r V_1^{(k-1)} - q_{r+1} V_1^{(k-2)} - \dots - q_n V_1^{(k-n+r-1)}, \quad k \in \mathbf{Z} \quad (30)$$

while iteration of (27) leads to

$$\begin{aligned} V_r^{(k)} &= -\frac{1}{q_n} \left( q_{r-1} V_n^{(k+1)} + \dots + q_1 V_n^{(k+r-1)} + V_n^{(k+r)} \right) \\ &= q_{r-1} V_1^{(k)} + \dots + q_1 V_1^{(k+r-2)} + V_1^{(k+r-1)}, \quad k \in \mathbf{Z}. \end{aligned} \quad (31)$$

## 5 Killing systems and related invariants for the Benenti class in Viète coordinates

The functions  $q_r(\lambda)$  defined in (24) can serve as a new set of variables on  $\mathcal{Q}$  (we will call them Viète coordinates). It turns out that these coordinates (that also reparametrize the infinite-dimensional manifold  $\mathcal{M}$  so that  $\mathcal{M} \ni u = (q_1(x), \dots, q_n(x))$  now) are much more convenient for our further purposes. The above considerations, in particular Lemma 2 and the corollary that follows, remain true independently of coordinate system since the Euler-Lagrange equations are invariant with respect to point transformations. In this section we sort out the structure of (18) and (21) for the Benenti class in Viète coordinates as well as we prove many other important relations.

The tensors  $L$ ,  $L^{-1}$ ,  $G^{(0)}$  and  $g^{(0)} = (G^{(0)})^{-1}$  have in Viète coordinates (24) the form:

$$L = \begin{pmatrix} -q_1 & 1 & & 0 \\ -q_2 & 0 & \ddots & \\ \vdots & & & 1 \\ -q_n & 0 & \cdots & 0 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 0 & \cdots & 0 & -\frac{1}{q_n} \\ 1 & 0 & 0 & -\frac{q_1}{q_n} \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -\frac{q_{n-1}}{q_n} \end{pmatrix}, \quad (32)$$

$$G^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \cdots & \cdots & q_1 \\ 0 & 1 & \cdots & \vdots \\ 1 & q_1 & \cdots & q_{n-1} \end{pmatrix}, \quad g^{(0)} = \begin{pmatrix} V_1^{(2n-2)} & \cdots & -q_1 & 1 \\ \cdots & \cdots & \cdots & 0 \\ -q_1 & 1 & \cdots & 0 \\ 1 & 0 & & 0 \end{pmatrix}. \quad (33)$$

so that  $L_j^i = V_i^{(n-j+1)}$  and  $g_{ij}^{(0)} = V_1^{(2n-i-j)}$ . Moreover, for the Benenti class, the system (18) attains in Viète coordinates (24) the form  $q_{t_r} = K_r(q)q_x$  or, explicitly

$$\frac{d}{dt_r} q_j = (q_{j+r-1})_x + \sum_{k=1}^{j-1} (q_k (q_{j+r-k-1})_x - q_{j+r-k-1} (q_k)_x) \equiv (Z_r^n [q])^j \quad r, j = 1, \dots, n \quad (34)$$

where  $q_\alpha = 0$  as soon as  $\alpha > n$  and  $(Z_r^n [q])^j$  denotes the  $j$ -th component of the vector field  $Z_r^n [q]$ . One proves (34) by a direct calculation, using (32) and (23). Observe, that the following symmetry relation takes place:  $(Z_i^n [q])^j = (Z_j^n [q])^i$ ,  $i, j = 1, \dots, n$ .

We can, in accordance with the above, also define the *infinite Killing hierarchy* for the Benenti class

$$\frac{d}{dt_r} q_j = (q_{j+r-1})_x + \sum_{k=1}^{j-1} (q_k (q_{j+r-k-1})_x - q_{j+r-k-1} (q_k)_x) \equiv (Z_r^\infty [q])^j \quad r, j = 1, \dots, \infty \quad (35)$$

that is formally given by the same expression as (34) but where we now do not impose the restriction  $q_\alpha = 0$  for  $\alpha > n$ . By comparing (34) and (35) one sees directly that for the  $r$ -th Killing vector field  $Z_r^n [q]$  from (34) its first  $n+1-r$  components coincide with the corresponding components of the infinite vector field  $Z_r^\infty [q]$ :

$$(Z_r^n [q])^j = (Z_r^\infty [q])^j \quad \text{for } j+r-1 \leq n. \quad (36)$$

**Lemma 4** *The infinite-component vector fields  $Z_r^\infty [q]$  in (35) mutually commute:*

$$[Z_i^\infty [q], Z_j^\infty [q]] = 0 \text{ for all } i, j = 1, \dots, \infty$$

**Proof.** This can be proved by using  $[Z_i^n [q], Z_j^n [q]] = 0$  for all  $i, j = 1, \dots, n$  and (36). Indeed, from (36) and the relation

$$(Z_i^\infty [q])^j = (Z_i^\infty)^j [q_1, \dots, q_{i+j-1}]$$



one finds that

$$([Z_i^\infty[q], Z_j^\infty[q]])^l = \left( [Z_i^{3(n-1)}[q], Z_j^{3(n-1)}[q]] \right)^l, \quad i, j, l = 1, \dots, n$$

for arbitrary  $n \in \mathbf{N}$ . ■

Let us point out that the infinite Killing hierarchy (35) is exactly the so-called universal hierarchy considered recently in [21],[22] from the point of view of Lax representation.

**Lemma 5** *In Viète coordinates the following relations hold:*

1.

$$\frac{\partial V_1^{(k)}}{\partial q_i} = \frac{\partial V_1^{(k+\alpha)}}{\partial q_{i+\alpha}} \text{ for } i = 1, \dots, n - \alpha, k \in \mathbf{Z} \quad (37)$$

2.

$$(L^k)_j^i = V_i^{(n+k-j)}, \quad k \in \mathbf{Z} \quad (38)$$

3.

$$g_{ij}^{(m)} = V_1^{(2n-m-i-j)}, \quad m \in \mathbf{Z}. \quad (39)$$

**Proof.** For relation (37) the proof is inductive with the help of formula (30). For relation (38) the proof is by induction with respect to  $k$ . By (32),  $L_j^i = V_i^{(n-j+1)}$ . By the induction assumption and due to the recursion (25)

$$(L^{k+1})_j^i = \sum_{r=1}^n L_r^i (L^k)_j^r = -q_i V_1^{(n+k-j)} + V_{i+1}^{(n+k-j)} = V_i^{(n+k-j+1)}$$

which concludes the inductive step up. Similarly, due to the recursion (27),

$$(L^{k-1})_j^i = \sum_{r=1}^n (L^{-1})_r^i (L^k)_j^r = V_{i-1}^{(n+k-j)} - \frac{q_{i-1}}{q_n} V_n^{(n+k-j)} = V_i^{(n+k-j-1)}.$$

which concludes the inductive step down. Finally, for relation (39), according to (33), we have  $g_{ij}^{(0)} = V_1^{(2n-i-j)}$ . By induction

$$g_{ij}^{(m+1)} = \sum_{k=1}^n V_1^{(2n-m-i-k)} (L^{-1})_j^k.$$

Thus, due to (32) we have for  $j < n$

$$g_{ij}^{(m+1)} = g_{i,j+1}^{(m)} = V_1^{(2n-m-i-j-1)}.$$

while for  $j = n$  we have

$$\begin{aligned} g_{in}^{(m+1)} &= -\frac{1}{q_n} \left( q_{n-1} V_1^{(n-m-i)} + \dots + q_1 V_1^{(2n-m-2-i)} + V_1^{(2n-m-1-i)} \right) \\ &= -\frac{1}{q_n} V_n^{(n-m-i)} = V_1^{(n-m-i-1)}, \end{aligned}$$

which follows from (29) and (31). This concludes the inductive step up. The induction down (for  $m < 0$ ) is proved in a similar way. ■

The next theorem describes symmetry properties of functions (19). Observe that due to (28) the functions (19) are in the Benenti case geodesic (without the potential part) for  $k = 0, \dots, n-1$ .

**Theorem 6** *For the Lagrangian densities*

$$\mathcal{L}^{n,m,k} = \frac{1}{2} \sum_{i,j=1}^n q_{i,x} g_{ij}^{(m)}(q) q_{j,x} - V_1^{(k)} = \frac{1}{2} \sum_{i,j=1}^n q_{i,x} V_1^{(2n-m-i-j)} q_{j,x} - V_1^{(k)}$$

*the following relations hold:*

1. for  $\alpha = 1, \dots, n-1$

$$E_i(\mathcal{L}^{n,m,k}) = E_{i-\alpha}(\mathcal{L}^{n,m+\alpha,k-\alpha}), \quad i = \alpha+1, \dots, n, \quad (40)$$

that can also be written as

$$E_i(\mathcal{L}^{n,m,k}) = E_{i+\alpha}(\mathcal{L}^{n,m-\alpha,k+\alpha}), \quad i = 1, \dots, n-\alpha. \quad (41)$$

2.

$$E_l(\mathcal{L}^{n,0,2n+\sigma}) = E_{l+1}(\mathcal{L}^{n+1,0,2n+\sigma+2}), \quad \sigma = 1, \dots, n-1, \quad l = \sigma+1, \dots, n. \quad (42)$$

3.

$$E_l(\mathcal{L}^{n,n-\sigma,0}) = E_l(\mathcal{L}^{n+1,n+1-\sigma,0}), \quad \sigma = 1, \dots, n-1, \quad l = 1, \dots, \sigma \quad (43)$$

$$E_{\sigma+l}(\mathcal{L}^{n,n-\sigma,-n})|_{q_j \rightarrow q_{j+1}} = E_l(\mathcal{L}^{n+1,n+1-\sigma,-n-1}), \quad l = 1, \dots, n-\sigma, \quad j = 1, \dots, n \quad (44)$$

The proof of this theorem can be found in Appendix B. As it will be shown in the next section, Theorem 6 guarantee that the form of invariants survives the passage from  $n$ -component to  $(n+1)$ -component Killing system and hence it will be crucial for the construction of soliton hierarchies. The index  $\sigma$  will be related with the number of components of the obtained soliton systems.

## 6 Elimination procedure

Using the results of the previous section we will now construct in a systematic way soliton hierarchies related to Benenti class of separation relations. These hierarchies will be generated by a procedure of elimination of variables in the set of dispersionless systems (34) with the help of Euler-Lagrange equations (21) (with suitable chosen parameters  $n$ ,  $m$  and  $k$  determining the metric tensor  $g^{(m)}$  and the separable potential  $V_1^{(k)}$ ). Actually, we present two separate elimination procedures, one for positive potentials (i.e. with  $k > n$ ) and one for negative potentials (i.e. those with  $k < 0$ ), leading to different soliton hierarchies.

As we pointed out, the set  $\mathcal{E} \subset \mathcal{M}$  of solutions of Euler-Lagrange equations (21) is invariant with respect to all the vector fields  $Z_r$  of Killing systems (18). The same must be true even in Viète coordinates: (21) written in Viète coordinates is also invariant with respect to (34). This means that we can use the set of equations (21) to eliminate some of the variables  $q_i$  in (34), since along the solutions of systems from (34) they are all the time mutually related by the relations (21).

### 6.1 Elimination for positive potentials

First, let us concentrate on the case with positive (polynomial) separable potentials. Our aim is to produce  $s$  ( $s \in \mathbf{N}$ ) commuting  $\sigma$ -component ( $\sigma \in \mathbf{N}$ ) vector fields (evolutionary systems) from (34) and (21). In order to do this we choose  $n$  as  $n = s + \sigma - 1$  and consider the set of systems (34) with this chosen  $n$ :

$$q_{t_r} = Z_r^{s+\sigma-1}[q_1, \dots, q_{s+\sigma-1}], \quad r = 1, \dots, n = s + \sigma - 1. \quad (45)$$

Notice that  $r$  can reach  $r = s + \sigma - 1$  but only up to  $r = s$  the first  $\sigma$  components of (45) are complete in the sense of the infinite Killing hierarchy (35), i.e. coincide with the corresponding components of this hierarchy (see (36)). We will now use (21) generated by  $\mathcal{L}^{n,0,2n+\sigma}$  in order to perform the elimination. The structure of these equations is described in the lemma below.

**Lemma 7** *The last  $n - \sigma$  invariant equations in (21) for  $\mathcal{L}^{n,0,2n+\sigma}$ , with  $n = s + \sigma - 1$  (so that  $m = 0$  and  $k = 2n + \sigma = 2s + 3\sigma - 2$ ) have the form*

$$\begin{aligned}
w_{\sigma+1}^{(n,\sigma)} &\equiv -2q_n + \varphi_{\sigma+1}^{(n,\sigma)}[q_1, \dots, q_{n-1}] = 0 \\
&\vdots \\
w_n^{(n,\sigma)} &\equiv -2q_{\sigma+1} + \varphi_n^{(n,\sigma)}[q_1, \dots, q_\sigma] = 0
\end{aligned} \tag{46}$$

where we denoted, to shorten the notation,  $E_i(\mathcal{L}^{n,0,2n+\sigma})$  as  $w_i^{(n,\sigma)}$ .

**Proof.** From the recursion (25) it follows that

$$V_1^{(n+j)} = V_1^{(n+j)}(q_1, \dots, q_{j+1}), \quad j = 0, \dots, n-1 \tag{47}$$

so that, again by (37) and (30)

$$\frac{\partial V_1^{(2n+\sigma)}}{\partial q_{n+1-j+\sigma}} = \frac{\partial V_1^{(n+j)}}{\partial q_1} = 2q_j + f_j(q_1, \dots, q_{j-1}) \quad j = 2, \dots, n, \quad \sigma = 1, \dots, n-1 \tag{48}$$

(for  $j = 1$  we would have  $f_1 \equiv 0$ ) where the first equality follows by inserting  $i = 1$ ,  $k = n + j$  and  $\alpha = n - j + \sigma$  in (37) and the second one from the fact that according to (28) and (30)  $V_1^{(n+j)} = -q_{j+1} + 2q_1q_j + \varphi_j(q_1, \dots, q_{j-1})$  for  $j = 2, \dots, n$ . On the other hand, for the geodesic Lagrangian density

$$\mathcal{L}^{n,0,0} = \frac{1}{2} \sum_{i,j=1}^n V_1^{(2n-i-j)} q_{i,x} q_{j,x}$$

from (B2), as  $V_1^{(k)} \neq 0$  for  $k \geq n-1$  and  $\partial V_1^{(k)}/\partial q_1 \neq 0$  for  $k \geq n$ , we find that

$$E_l(\mathcal{L}^{n,0,0}) = F_l[q_1, \dots, q_{n-l+1}], \quad l = 1, \dots, n.$$

Since  $\mathcal{L}_1^{n,0,2n+\sigma} = \mathcal{L}_1^{n,0,0} - V_1^{(2n+\sigma)}$ , we obtain (putting  $j = n+1-i+\sigma$  in (48))

$$E_i(\mathcal{L}^{n,0,2n+\sigma}) = -2q_{\sigma+n-i+1} + \varphi_i^{(n,\sigma)}[q_1, \dots, q_{\sigma+n-i}], \quad i = 1, \dots, n,$$

(where as usual we denote  $q_\alpha = 0$  for  $\alpha > n$ ) where

$$\varphi_i^{(n,\sigma)}[q_1, \dots, q_{\sigma+n-i}] = F_i[q_1, \dots, q_{n-i+1}] + f_{\sigma+n-i+1}(q_1, \dots, q_{\sigma+n-i}).$$

■

Due to their structure, equations (46) make it possible to successively express (eliminate) the variables  $q_{\sigma+1}, \dots, q_{s+\sigma-1} \equiv q_n$  as differential functions of  $q_1, \dots, q_\sigma$ :

$$\begin{aligned}
q_{\sigma+1} &= f_{\sigma+1}^n[q_1, \dots, q_\sigma] \\
q_{\sigma+2} &= f_{\sigma+2}^n[q_1, \dots, q_\sigma] \\
&\vdots \\
q_n &= f_n^n[q_1, \dots, q_\sigma].
\end{aligned} \tag{49}$$

Let us first observe that performing the elimination (49) in the systems (45) must lead to  $\sigma$ -component systems of the form  $\bar{q}_{t_r} = \bar{Z}_r^\sigma[q_1, \dots, q_\sigma]$ , while for each system in (45) the last  $s-1$  components turn into some system of differential consequences of  $w_1^{(n,\sigma)}, \dots, w_\sigma^{(n,\sigma)}$  (and are zero on  $\mathcal{S}_{u_0}$  i.e. they are satisfied along any solutions of  $\bar{q}_{t_r} = \bar{Z}_r^\sigma$ ). Therefore, after this elimination we obtain

$$\begin{cases} \bar{q}_{t_r} = \bar{Z}_r^\sigma[q_1, \dots, q_\sigma], \\ 0 = \varphi_r^i[w_1, \dots, w_\sigma], \quad i = \sigma+1, \dots, n \end{cases} \quad r = 1, \dots, n \tag{50}$$

with  $\bar{q} = (q_1, \dots, q_\sigma)^T$  and  $\varphi_r^i \equiv q_{i,t_r} - (Z_r^n)^i$ .

**Lemma 8** The first  $s$  vector fields  $\bar{Z}_r^\sigma$  in (50) commute:

$$[\bar{Z}_i^\sigma, \bar{Z}_j^\sigma] = 0, \quad i, j = 1, \dots, s.$$

**Proof.** Obviously, in general, for  $i, j = 1, \dots, n$ ,

$$[\overline{Z}_i^\sigma, \overline{Z}_j^\sigma] = V_{ij} [w_1^{(n,\sigma)}, \dots, w_\sigma^{(n,\sigma)}]$$

for some vector fields  $V_{ij}$  that vanish on  $\mathcal{E} \subset \mathcal{M}$  only. Assume for a moment that for  $n = s + \sigma - 1$  and for some  $i, j \leq s$  we have  $V_{ij} [w_1^{(n,\sigma)}, \dots, w_\sigma^{(n,\sigma)}] \neq 0$ . As the vector fields  $\overline{Z}_i^\sigma, \overline{Z}_j^\sigma$  were obtained by the reduction of the complete (in the sense of the infinite hierarchy) components of  $Z_i^n, Z_j^n$ , thus by increasing  $n \rightarrow n + \beta$ , we do not change the form of  $V_{ij}$ , which now has to be expressed by a higher dimension invariants  $w_j^{(n+\beta,\sigma)}$ :  $V_{ij} = V_{ij} [w_1^{(n+\beta,\sigma)}, \dots, w_\sigma^{(n+\beta,\sigma)}]$ . But  $w_j^{(n+\beta,\sigma)} = w_j^{(n+\beta,\sigma)}[q_1, \dots, q_{n+\beta}]$  and lower dimensional invariants  $w_j^{(n,\sigma)}$  are nonexpressible by the higher dimensional invariants  $w_i^{(n+\beta,\sigma)}$ , so we get a contradiction. ■

We will now show that this procedure leads in fact to an infinite hierarchy of commuting flows. In order to do this, we will for a moment introduce a new index so that the vector fields in (50) will be denoted  $\overline{Z}_r^{n,\sigma}$  as being obtained by reducing the  $n$ -component Killing systems (45).

**Lemma 9** *In the above notation*

$$\overline{Z}_r^{n+1,\sigma} = \overline{Z}_r^{n,\sigma} \text{ for } r = 1, \dots, s$$

**Proof.** According to (42) we have

$$w_{\sigma+i+1}^{(n+1,\sigma)} = w_{\sigma+i}^{(n,\sigma)} \text{ for } i = 1, \dots, n - \sigma.$$

Thus, increasing  $s$  to  $s + 1$  and keeping  $\sigma$  unaltered (so that  $n$  changes to  $n + 1$ ) the  $n - \sigma$  equations (49) change to  $n - \sigma + 1$  equations

$$\begin{aligned} q_{\sigma+i} &= f_{\sigma+i}^{n+1}[q_1, \dots, q_\sigma] = f_{\sigma+i}^n[q_1, \dots, q_\sigma], \quad i = 1, \dots, n - \sigma \\ q_{n+1} &= f_{n+1}^{n+1}[q_1, \dots, q_\sigma] \end{aligned}$$

so that the variables  $q_{\sigma+1}, \dots, q_n$  are expressed by the same functions of  $q_1, \dots, q_\sigma$  and a new elimination equation for  $q_{n+1}$  appears. Moreover,  $(Z_r^{n+1}[q])^j = (Z_r^n[q])^j$  for  $j = 1, \dots, \sigma$  and  $r = 1, \dots, s$ . Thus, replacing  $q_{\sigma+i}$  by  $f_{\sigma+i}^n[q_1, \dots, q_\sigma]$  in  $(Z_r^{n+1}[q])^j$  and in  $(Z_r^n[q])^j$  yields for  $j = 1, \dots, \sigma$  and  $r = 1, \dots, s$  the same expression. But the first operation leads to the reduced vector field  $\overline{Z}_r^{n+1,\sigma}$  while the second - to  $\overline{Z}_r^{n,\sigma}$ . ■

Let us now take  $s + 1$  instead of  $s$  (so that  $n \rightarrow n + 1$ ) in (45) and (49) and perform the reduction. According to the above lemma we obtain the following sequence of  $s + 1$  reduced systems:

$$\overline{q}_{t_r} = \overline{Z}_r^{n+1,\sigma} = \overline{Z}_r^{n,\sigma} \text{ for } r = 1, \dots, s \text{ and } \overline{q}_{t_{n+1}} = \overline{Z}_{n+1}^{n+1,\sigma}$$

i.e. we obtain the same sequence of  $s$  systems as before plus an additional system at the end of the sequence. Therefore, we see that this procedure leads to infinite hierarchies of commuting systems, since we can always increase  $n$  as much as we please without altering the already obtained systems generated in previous steps.

The procedure described above can be generalized by using only some part of the equations in (46) in order to perform the elimination, since all of these equations are invariant along the flows of our Killing systems. Namely, we can skip the last  $\alpha$  (with  $0 \leq \alpha \leq n - \sigma - 1 = s - 2$ ) equations in (46) and use only the remaining equations (i.e.  $w_{\sigma+1}^{(n,\sigma)} = 0, w_{\sigma+2}^{(n,\sigma)} = 0, \dots, w_{n-\alpha}^{(n,\sigma)} = 0$ ) to eliminate  $q_{\sigma+\alpha+1}, \dots, q_n$  in the Killing systems with  $n = s + \sigma + \alpha - 1$ :

$$q_{t_r} = Z_r^{s+\sigma+\alpha-1} [q_1, \dots, q_{s+\sigma+\alpha-1}], \quad r = 1, \dots, n = s + \sigma + \alpha - 1. \quad (51)$$

Thus, the index  $\alpha$  indicates how many of the last equations in (46) we "forget" about. It turns out that the elimination that follows leads also to hierarchies of commuting equations. To see that, let us first

observe, that this elimination can formally be obtained by performing the above described procedure with the help of the Lagrangian density  $\mathcal{L}^{n,-\alpha,2n+\sigma+\alpha}$ , since according to Theorem 6 we have

$$E_i(\mathcal{L}^{n,-\alpha,2n+\sigma+\alpha}) = E_{i-\alpha}(\mathcal{L}^{n,0,2n+\sigma}) \equiv w_{i-\alpha}^{(n,\sigma)} \quad \text{for } i = \alpha + 1, \dots, n. \quad (52)$$

Denoting  $E_i(\mathcal{L}^{n,m,2n+k})$  as  $w_i^{(n,m,k)}$ , where now  $w_i^{(n,0,k)} \equiv w_i^{(n,k)}$  (in the notation of (46)), the last  $n - \sigma - \alpha$  Euler-Lagrange equations (invariants), associated with  $\mathcal{L}_1^{n,-\alpha,2n+\sigma+\alpha}$ , have therefore the form

$$\begin{aligned} w_{\sigma+\alpha}^{(n,-\alpha,\sigma+\alpha)} &= w_{\sigma}^{(n,\sigma)} = w_{\sigma}^{(n,\sigma)}[q_1, \dots, q_n] = 0 \\ w_{\sigma+1+\alpha}^{(n,-\alpha,\sigma+\alpha)} &= w_{\sigma+1}^{(n,\sigma)} = -2q_n + \varphi_{\sigma+1}^{(n,\sigma)}[q_1, \dots, q_{n-1}] = 0 \\ &\vdots \\ w_n^{(n,-\alpha,\sigma+\alpha)} &= w_{n-\alpha}^{(n,\sigma)} = -2q_{\sigma+\alpha+1} + \varphi_{n-\alpha}^{(n,\sigma)}[q_1, \dots, q_{\sigma+\alpha}] = 0. \end{aligned} \quad (53)$$

These equations make it possible to successively express (eliminate) the variables  $q_{\sigma+\alpha+1}, \dots, q_n$  as differential functions of  $q_1, \dots, q_{\sigma+\alpha}$ , which yields

$$\begin{aligned} q_{\sigma+\alpha+1} &= q_{\sigma+\alpha+1}[q_1, \dots, q_{\sigma+\alpha}] \\ &\vdots \\ q_n &= q_n[q_1, \dots, q_{\sigma+\alpha}]. \end{aligned} \quad (54)$$

Therefore, after this elimination the Killing equations (51) take the form

$$\begin{cases} \bar{q}_{t_r} = \bar{Z}_r^{\sigma+\alpha}[\bar{q}], \\ 0 = \varphi_r^i \left[ w_1^{(n,-\alpha,\sigma+\alpha)}, \dots, w_{\sigma+\alpha}^{(n,-\alpha,\sigma+\alpha)} \right], \quad i = \sigma + \alpha + 1, \dots, n \end{cases} \quad r = 1, \dots, n = s + \sigma + \alpha - 1 \quad (55)$$

with  $\bar{q} = (q_1, \dots, q_{\sigma+\alpha})^T$  and  $\varphi_r^i \equiv q_{i,t_r} - (Z_r^n)^i$  (so that the reduced systems will have  $N = \sigma + \alpha$  components). Similarly as before, in Killing equations (51), only up to  $r = s$  the first  $\sigma + \alpha$  components are complete in the sense of the infinite Killing hierarchy (35). As before, it stems from the fact that the first  $s$  vector fields  $\bar{Z}_r^{\sigma+\alpha}$  commute to zero:

$$[\bar{Z}_i^{\sigma+\alpha}, \bar{Z}_j^{\sigma+\alpha}] = 0, \quad i, j = 1, \dots, s.$$

the proof of which is analogous as in the case  $\alpha = 0$  but now we have to take  $n = s + \sigma + \alpha - 1$ . As previously, we can repeat the elimination procedure taking  $s + 1$  instead of  $s$  (so that  $n$  increases to  $n + 1$  and  $k = 2n + \sigma + \alpha$  increases to  $2(n + 1) + \sigma + \alpha = k + 2$  while  $\sigma$  and  $\alpha$  are kept unaltered). By the same argument as before, this new procedure (with  $n + 1$  instead of  $n$ ) will lead to a sequence of  $s + 1$  autonomous  $(\sigma + \alpha)$ -component systems in which the first  $s$  systems will coincide with the corresponding systems obtained from the original procedure (with  $n$ ). Thus, again we will obtain infinite hierarchies of soliton systems.

## 6.2 Elimination for negative potentials

We now present the second possibility of elimination - with the use of negative (rational) separable potentials. Again, our aim is to produce  $s$  ( $s \in \mathbf{N}$ ) commuting  $\sigma$ -component ( $\sigma \in \mathbf{N}$ ) vector fields (evolutionary systems) from (34) and (21). This time however we have to choose  $n = s + 2\sigma - 1$  and the Lagrangian density  $\mathcal{L}^{n,n-\sigma,-n}$  in order to create an infinite hierarchy of commuting flows.

**Lemma 10** *The first  $n - \sigma$  invariant equations (21) with  $\mathcal{L}^{n,n-\sigma,-n}$ , i.e. with  $m = n - \sigma$  and  $k = -n$ , have the form*

$$\begin{aligned} v_1^{(n,\sigma)} &\equiv -\frac{1}{q_n^2} + \gamma_1^{(n,\sigma)}[q_1, \dots, q_\sigma] = 0, \\ v_i^{(n,\sigma)} &\equiv \frac{2q_{n-i+1}}{q_n^3} + \gamma_i^{(n,\sigma)}[q_1, \dots, q_{\sigma-i+1}, q_{n-i+2}, \dots, q_n] = 0, \quad i = 2, \dots, n - \sigma. \end{aligned} \quad (56)$$

where we denote, to shorten the notation,  $E_i(\mathcal{L}^{n,n-\sigma,-n})$  as  $v_i^{(n,\sigma)}$  and  $q_\alpha = 0$  when  $\alpha < 1$ .

**Proof.** From the recursion (27) it follows that

$$V_1^{(-j)} = V_1^{(-j)}(q_{n-j+1}, \dots, q_n), \quad j = 1, \dots, n \quad (57)$$

From this and from (B5), (47) and (B7), we have

$$E_i(\mathcal{L}^{n, n-\sigma, 0}) = G_i[q_1, \dots, q_{\sigma-i+1}], \quad i = 1, \dots, \sigma, \\ E_i(\mathcal{L}^{n, n-\sigma, 0}) = G_i[q_{n-i+1+\sigma}, \dots, q_n], \quad i = \sigma + 1, \dots, n - \sigma. \quad (58)$$

Moreover, by using Lemma 5 we find

$$\frac{\partial V_1^{(-n+1-i)}}{\partial q_1} = -\frac{2q_{n-i+1}}{q_n^3} + g_i(q_{n-i+2}, \dots, q_n), \quad i = 2, \dots, n - \sigma$$

and

$$\frac{\partial V_1^{(-n+\sigma)}}{\partial q_{\sigma+1}} = \frac{\partial V_1^{(-n)}}{\partial q_1} = \frac{1}{q_n^2}.$$

Plugging all this into  $E_i(\mathcal{L}^{n, n-\sigma, -n})$ ,  $i = 1, \dots, n - \sigma$ , we obtain (56) where  $\gamma_i^{(n, \sigma)}[q] = G_i[q] - g_i(q)$ . ■

Let us now consider the following Killing systems

$$q_{t_r} = Z_r^n[q], \quad r = \sigma + 1, \dots, \sigma + s, \quad \text{with } n = s + 2\sigma - 1 \quad (59)$$

We can use the  $n - \sigma$  equations (56) to successively express (eliminate) the variables  $q_{\sigma+1}, \dots, q_n$  as differential functions of  $q_1, \dots, q_\sigma$ . This leads to the elimination relations of the form

$$q_{\sigma+i} = f_{\sigma+i}^n[q_1, \dots, q_\sigma], \quad i = 1, \dots, n - \sigma. \quad (60)$$

Performing the elimination (60) in (59) we obtain an autonomous sequence of  $s$  evolution equations

$$\bar{q}_{t_r} = \bar{Z}_r^\sigma[q_1, \dots, q_\sigma], \quad r = \sigma + 1, \dots, \sigma + s \quad (61)$$

such that the vector fields  $\bar{Z}_r^\sigma$  mutually commute to zero. One proves this by the same arguments as in the positive case, since the first  $\sigma$  components of all the vector fields in (59) are complete in the sense of infinite hierarchy (35)).

Analogously to the positive case, we will now show that this procedure leads to an infinite hierarchy. As in the positive case, we will for a moment introduce a new index so that the vector fields in (61) will be denoted  $\bar{Z}_r^{n, \sigma}$  as being obtained by reducing the  $n$ -component Killing systems (59).

**Lemma 11** *In the notation as above*

$$\bar{Z}_{r+1}^{n+1, \sigma} = \bar{Z}_r^{n, \sigma} \text{ for } r = \sigma + 1, \dots, \sigma + s$$

**Proof.** Let us observe that, according to results (41), (43), (44) and (B9)

$$v_i^{(n+1, \sigma)} = v_i^{(n, \sigma)} \Big|_{q_j \rightarrow q_{j+1}, \quad j=n-i+1, \dots, n} \quad \text{for } i = 1, \dots, n - \sigma. \quad (62)$$

Thus, increasing  $s$  to  $s + 1$  and keeping  $\sigma$  unaltered (so that  $n$  changes to  $n + 1$ ) the  $n - \sigma$  equations (60) change to  $n - \sigma + 1$  equations

$$q_{\sigma+1} = f_{\sigma+1}^{n+1}[q_1, \dots, q_\sigma], \quad q_{\sigma+i+1} = f_{\sigma+i+1}^{n+1}[q_1, \dots, q_\sigma] = f_{\sigma+i}^n[q_1, \dots, q_\sigma], \quad i = 1, \dots, n - \sigma. \quad (63)$$

Observe that the last  $n - \sigma$  equations (60) express now  $q_{\sigma+i+1}$  (instead of  $q_{\sigma+i}$ ) as  $f_{\sigma+i}^n[q_1, \dots, q_\sigma]$ . On the other hand, due to (34), by changing  $q_i \rightarrow q_{i+1}$  for all  $i > \sigma$  in the sequence  $q_{t_r} = Z_r^{n+1}[q]$  we transform it so that  $(Z_r^{n+1}[q])^j \rightarrow (Z_{r+1}^{n+1}[q])^j$  for  $j = 1, \dots, \sigma$  and  $r = \sigma + 1, \dots, \sigma + s$ . Thus, inserting  $f_{\sigma+i}^n[q_1, \dots, q_\sigma]$

instead of  $q_{\sigma+i+1}$  in  $(Z_{r+1}^{n+1}[q])^j$  (for  $j = 1, \dots, \sigma$  and  $r = \sigma + 1, \dots, \sigma + s$ ) yields the same expression as inserting the same function  $f_{\sigma+i}^n[q_1, \dots, q_\sigma]$  instead of  $q_{\sigma+i}$  in  $(Z_r^{n+1}[q])^j$ . But the first operation leads to the reduced vector field  $\overline{Z}_{r+1}^{n+1, \sigma}$  while the second - to  $\overline{Z}_r^{n, \sigma}$ . ■

Let us now take  $s + 1$  instead of  $s$  (so that  $n \rightarrow n + 1$ ) in (59) and (60) and perform the reduction. According to Lemma 11 we obtain the following sequence of  $s + 1$  reduced systems:

$$\overline{q}_{t_{\sigma+1}} = \overline{Z}_{\sigma+1}^{n+1, \sigma}, \quad \overline{q}_{t_{r+1}} = \overline{Z}_{r+1}^{n+1, \sigma} = \overline{Z}_r^{n, \sigma} \text{ for } r = \sigma + 1, \dots, \sigma + s$$

i.e. we obtain the same sequence of  $s$  systems as before but shifted and an additional system in the beginning of the sequence. This first system can therefore be treated as a next system in some infinite, commuting hierarchy of vector fields.

Let us also observe that we could use (60) to eliminate variables in Killing systems of the form (45) and this would lead to a system of  $s$  commuting evolutionary systems. However, this choice does not lead to any hierarchy: by increasing  $s$  to  $s + 1$  we obtain a different sequence of systems.

As before, this procedure can be generalized: we can use the first  $n - \sigma - \alpha$  equations ( $0 \leq \alpha < n - \sigma - 1$ ) in (56) to eliminate  $q_{\sigma+\alpha+1}, \dots, q_n$  from the following sequence of  $s$  Killing systems

$$q_{t_r} = Z_r^n[q], \quad r = \sigma + \alpha + 1, \dots, \sigma + \alpha + s \text{ with } n = s + 2(\sigma + \alpha) - 1. \quad (64)$$

This elimination leads - similarly as above - to  $s$  commuting to zero  $N = \sigma + \alpha$ -component systems  $\overline{q}_{t_r} = \overline{Z}_r^{\sigma+\alpha}[q_1, \dots, q_{\sigma+\alpha}]$  and by increasing  $s$  by 1 we always obtain a new system of the hierarchy at the beginning of the sequence.

Next section contains some examples of the above described elimination procedures.

## 7 Examples

### 7.1 Elimination with positive potentials

Below we will present some examples performed with the help of the (generalized) procedure described in the previous section. Soliton hierarchies are now classified by pairs  $(\sigma, \alpha)$ ,  $\sigma = 1, 2, \dots, \alpha = 0, 1, \dots$ , where  $N = \sigma + \alpha$  is a number of components in the systems of a given hierarchy. Assume we would like to construct first  $s$  members of the hierarchy. We have then to fix  $n = s + \sigma + \alpha - 1$  and take first  $s$  Killing equations in (51). Then, we have to eliminate coordinates  $q_{\sigma+\alpha+1}, \dots, q_{\sigma+\alpha+s-1} = q_n$  using invariants  $w_{\sigma+1}^{(n, \sigma)} = 0, \dots, w_{n-\alpha}^{(n, \sigma)} = 0$ . According to (52), these invariants can be generated, for example, from  $\mathcal{L}^{n, 0, 2n+\sigma}$  by taking the equations  $E_i(\mathcal{L}^{n, 0, 2n+\sigma}) = 0$ ,  $i = \sigma + 1, \dots, n - \alpha$ . After the elimination procedure, soliton equations are represented by first  $N = \sigma + \alpha$  components of first  $s$  reduced Killing equations. Observe, that in this procedure the first soliton equation has always the trivial form  $\overline{q}_{t_1} = \overline{q}_x$ ,  $\overline{q} = (q_1, \dots, q_{\sigma+\alpha})^T$ .

Let us start with a one-field hierarchy:  $N = \sigma + \alpha = 1$ . There is only one possibility here:  $\sigma = 1, \alpha = 0$ . We present how to produce first  $s = 3$  flows which will be recognized as the first members of the KdV hierarchy. We have therefore to take  $n = 3$  and  $k = 7$ . Killing systems (34) have the form:

$$\begin{aligned} \frac{d}{dt_1} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \begin{pmatrix} q_{1,x} \\ q_{2,x} \\ q_{3,x} \end{pmatrix} = Z_1^3 \\ \frac{d}{dt_2} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \begin{pmatrix} q_{2,x} \\ q_{3,x} + q_1 q_{2,x} - q_2 q_{1,x} \\ q_1 q_{3,x} - q_3 q_{1,x} \end{pmatrix} = Z_2^3 \\ \frac{d}{dt_3} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \begin{pmatrix} q_{3,x} \\ q_1 q_{3,x} - q_3 q_{1,x} \\ q_2 q_{3,x} - q_3 q_{2,x} \end{pmatrix} = Z_3^3 \end{aligned} \quad (65)$$

while the Lagrangian  $\mathcal{L}^{3, 0, 7}$  is

$$\mathcal{L}^{3, 0, 7} = \frac{1}{2} q_1^2 q_{1,x}^2 - \frac{1}{2} q_2 q_{1,x}^2 - q_1 q_{1,x} q_{2,x} + q_{1,x} q_{3,x} + \frac{1}{2} q_{2,x}^2 - 2 q_2 q_3 + 3 q_1^2 q_3 + 3 q_1 q_2^2 - 4 q_1^3 q_2 + q_1^5$$

and the Euler-Lagrange equations (46) for the above Lagrangian attain the form

$$\begin{aligned} w_2^{(3,1)} &\equiv -2q_3 + 6q_1q_2 - 4q_1^3 + \frac{1}{2}q_{1,x}^2 + q_1q_{1,xx} - q_{2,xx} = 0 \\ w_3^{(3,1)} &\equiv -2q_2 + 3q_1^2 - q_{1,xx} = 0. \end{aligned}$$

These equations can be solved with respect to  $q_2, q_3$  yielding (49) of the form

$$q_2 = -\frac{1}{2}q_{1,xx} + \frac{3}{2}q_1^2, \quad q_3 = \frac{1}{4}q_{1,xxxx} - \frac{5}{2}q_1q_{1,xx} - \frac{5}{4}q_{1,x}^2 + \frac{5}{2}q_1^3. \quad (66)$$

Substituting it to the above Killing systems gives (50) that read now explicitly as

$$\begin{cases} q_{1,t_1} = q_{1,x} = \overline{Z}_1^1 \\ 0 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} q_{1,t_2} = -\frac{1}{2}q_{1,xxx} + 3q_1q_{1,x} = \overline{Z}_2^1 \\ 0 = 0 \\ 0 = \frac{1}{2}w_{1,x} \end{cases} \\ \begin{cases} q_{1,t_3} = \frac{1}{4}q_{1,xxxx} - 5q_{1,x}q_{1,xx} - \frac{5}{2}q_1q_{1,xxx} + \frac{15}{2}q_1^2q_{1,x} = \overline{Z}_3^1 \\ 0 = \frac{1}{2}w_{1,x} \\ 0 = -\frac{1}{4}w_{1,xxx} + \frac{3}{2}q_1w_{1,x} \end{cases} \end{cases}$$

so that the first components  $q_{1,t_i} = \overline{Z}_i^1[q_1]$  are the first three flows of the KdV hierarchy while the remaining equations are just differential consequences of  $w_1$ , which of course vanish on any  $\mathcal{S}_{u_0}$ . By taking larger  $s$  we can produce an arbitrary number of flows from the KdV hierarchy.

Next, let us consider two-field systems:  $N = \sigma + \alpha = 2$ . There are two possibilities:  $(\sigma, \alpha) = (2, 0)$  and  $(\sigma, \alpha) = (1, 1)$ . Therefore, as a second example we consider the case  $(\sigma, \alpha) = (2, 0)$ , and  $s = 3$ . We have now to take  $n = s + \sigma + \alpha - 1 = 4$  and  $k = 2n + \sigma = 10$ . The Euler-Lagrange equations  $E_4(\mathcal{L}^{4,0,10}) = w_4^{(4,2)} = 0$  and  $E_3(\mathcal{L}^{4,0,10}) = w_3^{(4,2)} = 0$  can be solved with respect to  $q_3, q_4$  yielding (49) of the form

$$\begin{aligned} q_3 &= -\frac{1}{2}q_{1,xx} + 3q_1q_2 - 2q_1^3 \\ q_4 &= \frac{1}{4}q_{1,x}^2 - \frac{1}{2}q_{2,xx} - q_1q_{1,xx} - \frac{7}{2}q_1^4 + 3q_1^2q_2 + \frac{3}{2}q_2^2. \end{aligned}$$

Substituting it to two first components of Killing equations  $Z_2^4[q], Z_3^4[q]$  yields two first nontrivial members of another two-field soliton hierarchy:

$$\begin{aligned} q_{1,t_2} &= q_{2,x} \\ q_{2,t_2} &= -\frac{1}{2}q_{1,xxx} + 4q_1q_{2,x} + 2q_2q_{1,x} - 6q_1^2q_{1,x}, \end{aligned} \quad (67)$$

and

$$\begin{aligned} q_{1,t_3} &= -\frac{1}{2}q_{1,xxx} + 3q_1q_{2,x} + 3q_2q_{1,x} - 6q_1^2q_{1,x} \\ q_{2,t_3} &= -\frac{1}{2}q_{2,xxx} - \frac{3}{2}q_1q_{1,xxx} + 3q_2q_{2,x} + 6q_1q_2q_{1,x} + 6q_1^2q_{2,x} - 18q_1^3q_{1,x}. \end{aligned}$$

In the second two-field case  $(\sigma, \alpha) = (1, 1)$ , if we keep  $s = 3$  unchanged, we have to take  $n = s + \sigma + \alpha - 1 = 4$  and  $k = 2n + \sigma = 9$ . From the Euler-Lagrange equations (46) for  $\mathcal{L}^{4,0,9}$

$$E_3(\mathcal{L}^{4,0,9}) = w_3^{(4,1)} = 0, \quad E_2(\mathcal{L}^{4,0,9}) = w_2^{(4,1)} = 0$$

we can eliminate  $q_3$  and  $q_4$ , which yields (54). Explicitly, we obtain:

$$\begin{aligned} q_3 &= \frac{1}{4}q_{1,xxxx} - \frac{5}{2}q_1q_{1,xx} - \frac{5}{4}q_{1,x}^2 + \frac{5}{2}q_1^3 \\ q_4 &= \frac{1}{4}q_{2,xxx} - \frac{1}{4}q_1q_{1,xxx} - \frac{3}{4}q_{1,x}q_{1,xx} - \frac{1}{2}q_{1,xx}^2 - q_2q_{1,xx} - \frac{5}{2}q_{1,x}q_{2,x} \\ &\quad + 4q_1^2q_{1,xx} - \frac{5}{2}q_1q_{2,xx} + \frac{25}{4}q_1q_{1,x}^2 + 3q_1^2q_2 - \frac{7}{2}q_1^4 + \frac{3}{2}q_2^2. \end{aligned}$$

Then, two first components of the Killing equations  $q_{t_2} = Z_2^4[q], q_{t_3} = Z_3^4[q]$  turn into

$$\begin{aligned} q_{1,t_2} &= q_{2,x} \\ q_{2,t_2} &= -\frac{1}{2}q_{2,xxx} + \frac{1}{2}q_1q_{1,xxx} + q_{1,x}q_{1,xx} + 4q_1q_{2,x} + 2q_2q_{1,x} - 6q_1^2q_{1,x}, \end{aligned}$$



and

$$\begin{aligned} q_{1,t_3} &= -\frac{1}{2}q_{2,xxx} + \frac{1}{2}q_1q_{1,xxx} + q_{1,x}q_{1,xx} + 3q_1q_{2,x} + 3q_2q_{1,x} - 6q_1^2q_{1,x} \\ q_{2,t_3} &= \frac{1}{4}q_{2,xxxxx} - \frac{1}{4}q_1q_{1,xxxxx} - q_{1,x}q_{1,xxxx} - \frac{7}{4}q_{1,xx}q_{1,xxx} - q_2q_{1,xxx} - \frac{7}{2}q_{2,x}q_{1,xx} \\ &\quad + \frac{9}{2}q_1^2q_{1,xx} - 3q_1q_{2,xxx} - \frac{9}{2}q_{1,x}q_{2,xx} + 21q_1q_{1,x}q_{1,xx} + 6q_1q_2q_{1,x} + 3q_2q_{2,x} \\ &\quad + 6q_1^2q_{2,x} + 6q_{1,x}^2 - 18q_1^3q_{1,x}. \end{aligned}$$

Finally, we shortly mention the three-field case:  $N = \sigma + \alpha = 3$ . There are three different hierarchies with the following first nontrivial member of each hierarchy:

for  $(\sigma, \alpha) = (3, 0)$ :

$$\begin{aligned} q_{1,t_2} &= q_{2,x} \\ q_{2,t_2} &= q_{3,x} + q_1q_{2,x} - q_2q_{1,x} \\ q_{3,t_2} &= -\frac{1}{2}q_{2,xxx} - 12q_1q_2q_{1,x} - 6q_1^2q_{2,x} + 3q_2q_{2,x} + 2q_3q_{1,x} + 4q_1q_{3,x} + 10q_1^3q_{1,x}, \end{aligned}$$

for  $(\sigma, \alpha) = (2, 1)$ :

$$\begin{aligned} q_{1,t_2} &= q_{2,x} \\ q_{2,t_2} &= q_{3,x} + q_1q_{2,x} - q_2q_{1,x} \\ q_{3,t_2} &= -\frac{1}{2}q_{2,xxx} + \frac{1}{2}q_1q_{1,xxx} + q_{1,x}q_{1,xx} - 12q_1q_2q_{1,x} - 6q_1^2q_{2,x} + 3q_2q_{2,x} \\ &\quad + 2q_3q_{1,x} + 4q_1q_{3,x} + 10q_1^3q_{1,x}, \end{aligned}$$

and for  $(\sigma + \alpha) = (1, 2)$ :

$$\begin{aligned} q_{1,t_2} &= q_{2,x} \\ q_{2,t_2} &= q_{3,x} + q_1q_{2,x} - q_2q_{1,x} \\ q_{3,t_2} &= -\frac{1}{2}q_{3,xxx} + \frac{1}{2}q_1q_{2,xxx} + \frac{1}{2}q_2q_{1,xxx} - \frac{1}{2}q_1^2q_{1,xxx} - 2q_1q_{1,x}q_{1,xx} + q_{1,x}q_{2,xx} + q_{2,x}q_{1,xx} \\ &\quad - \frac{1}{2}q_{1,x}^3 + 2q_3q_{1,x} + 4q_1q_{3,x} - 12q_1q_2q_{1,x} - 6q_1^2q_{2,x} + 3q_2q_{2,x} + 10q_1^3q_{1,x} \end{aligned}$$

In general, for a fixed  $N = \sigma + \alpha$ , this procedure leads to  $N$  different  $N$ -component hierarchies of soliton systems. As the field representation of constructed hierarchies is non-standard, it is not easy to recognize which hierarchies are known and which are new. We immediately recognized the KdV hierarchy. We also found that two-field hierarchy starting from (67) turns after the transformation

$$u_1 = -3q_1^2 + 2q_2, \quad u_2 = 2q_1, \quad x \rightarrow \sqrt{2}ix, \quad t \rightarrow \sqrt{2}it$$

into the 2-component coupled KdV hierarchy in the representation of Fordy and Antonowicz [23]. For example, the first flow of this hierarchy (67) turns into

$$\begin{aligned} u_{1,t_1} &= \frac{1}{4}u_{2,xxx} + \frac{1}{2}u_2u_{1,x} + u_1u_{2,x} \\ u_{2,t_1} &= u_{1,x} + \frac{3}{2}u_2u_{2,x}. \end{aligned}$$

(yielding (3.18) in [23]).

## 7.2 Elimination with negative potentials

We start by presenting the first two ( $s = 2$ ) flows of the only  $N = 1$ -component hierarchy that can be obtained within our scheme by using the negative separable potentials. Since  $N = 1 = \alpha + \sigma$ , the only choice is to put  $\sigma = 1$ ,  $\alpha = 0$ , which yields  $n = s + 2(\sigma + \alpha) - 1 = 3$ . The Euler-Lagrange equations (56) for

$$\mathcal{L}^{n,n-\sigma,-n} = \mathcal{L}^{3,2,-3} = \frac{1}{2}q_{1,x}^2 - \frac{q_{2,x}q_{3,x}}{q_3} + \frac{q_{3,x}^2q_2}{2q_3} - \frac{q_1}{q_3^2} + \frac{q_2^2}{q_3^3}$$

attain the form:

$$-1 - q_3^2q_{1,xx} = 0, \quad 4q_2 + 2q_3^2q_{3,xx} - q_3q_{3,x}^2 = 0 \quad (68)$$

which allows for expressing  $q_2$  and  $q_3$  as differential functions of  $q_1$ :

$$\begin{aligned} q_3 &= q_3[q_1] = (-q_{1,xx})^{-1/2} \\ q_2 &= q_2[q_1] = -\frac{1}{16} (5q_{1,xxx}^2 - 4q_{1,xx}q_{1,xxxx}) (-q_{1,xx})^{-7/2} \end{aligned}$$

(here and in what follows we only consider the positive solution for  $q_n$ , otherwise we can change  $t \rightarrow -t$ ). Substituting these expressions to the Killing systems (64) and performing the necessary derivations we obtain the following two commuting flows:

$$q_{1,t_2} = (q_2[q_1])_x \quad , \quad q_{1,t_3} = (q_3[q_1])_x$$

with the differential functions  $q_2[q_1]$  and  $q_3[q_1]$  given as above. After substitution  $u = -q_{1,xx}$  the second equation turns into the well known Harry Dym equation while the first one becomes the second member of the hierarchy. If we want to produce a next member of this hierarchy we have to take  $s = 3$ . According to the general remarks in the previous section this new system will appear as the first system in our sequence of systems.

Let us now consider two-field systems:  $N = \sigma + \alpha = 2$ . As before, we have now two choices:  $(\sigma, \alpha) = (1, 1)$  and  $(\sigma, \alpha) = (2, 0)$ . We start with  $(\sigma, \alpha) = (2, 0)$ . We have now  $n = s + 2\sigma - 1 = 5$  and thus we consider the Lagrangian  $\mathcal{L}^{n,n-\sigma,-n} = \mathcal{L}^{5,3,-5}$ . The associated Euler-Lagrange equations (56) can be written as

$$\begin{aligned} q_5^2 (q_{1,x}^2 + 2q_{1,xx}q_1 - 2q_{2,xx}) - 2 &= 0, \quad 2q_4 - q_5^3 q_{1,xx} = 0, \\ 4q_3q_5 - q_{5,x}^2 q_5^2 - 6q_4^2 + 2q_5^3 q_{5,xx} &= 0 \end{aligned}$$

and they can be solved to

$$\begin{aligned} q_5 &= f_5^5[q_1, q_2] = 2w^{-1/2} \\ q_4 &= f_4^5[q_1, q_2] = 4q_{1,xx}w^{-3/2} \\ q_3 &= f_3^5[q_1, q_2] = \left(-\frac{5}{2}w_x^2 + 12q_{1,xx}w + 2ww_{xx}\right)w^{-7/2} \end{aligned}$$

where  $w = 2q_{1,x}^2 + 4q_{1,xx}q_1 - 4q_{2,xx}$ . Substituting it into (64) we arrive at the following two commuting two-component systems:

$$\begin{aligned} q_{1,t_3} &= (f_3^5[q_1, q_2])_x \\ q_{2,t_3} &= q_1 (f_3^5[q_1, q_2])_x - (f_3^5[q_1, q_2]) q_{1,x} + (f_4^5[q_1, q_2])_x \end{aligned}$$

and

$$\begin{aligned} q_{1,t_4} &= (f_4^5[q_1, q_2])_x \\ q_{2,t_4} &= q_1 (f_4^5[q_1, q_2])_x - (f_4^5[q_1, q_2]) q_{1,x} + (f_5^5[q_1, q_2])_x \end{aligned} \tag{69}$$

The system (69) can be written more explicitly as

$$\begin{aligned} q_{1,t_4} &= 2(2q_{1,xxx}w - 3q_{1,xx}w_x)w^{-5/2} \\ q_{2,t_4} &= (4q_1q_{1,xxx}w - 6q_1q_{1,xx}w_x - 4q_{1,x}q_{1,xx}w - ww_{xx})w^{-5/2}. \end{aligned}$$

Finally, let us consider the case  $(\sigma, \alpha) = (1, 1)$ . Again, we have  $n = 5$ , but this time we consider the Lagrangian  $\mathcal{L}^{n,n-\sigma-\alpha,-n+\alpha} = \mathcal{L}^{5,3,-4}$ . Its first  $n - \sigma - \alpha = 3$  Euler-Lagrange equations

$$\begin{aligned} 0 &= -q_5^2 q_{1,xx} - 1 \\ 0 &= 4q_4 + 2q_5^2 q_{5,xx} - q_5 q_{5,x}^2 \\ 0 &= 2q_3q_5 - q_5^2 q_{4,x}q_{5,x} + q_4q_5q_{5,x}^2 - 3q_4^2 + q_5^3 q_{4,xx} - q_5^2 q_4q_{5,xx} \end{aligned}$$

yield the following elimination equations:

$$\begin{aligned} q_5 &= f_5^5[q_1] = (-q_{1,xx})^{-1/2} \\ q_4 &= f_4^5[q_1] = -\frac{1}{2^4} (5q_{1,xxx}^2 - 4q_{1,xx}q_{1,xxxx}) (-q_{1,xx})^{-7/2} \\ q_3 &= f_3^5[q_1] = \frac{1}{2^9} P[q_1] (-q_{1,xx})^{-13/2} \end{aligned} \quad (70)$$

where  $P[q_1]$  is some complicated differential polynomial of  $q_1$  (homogeneous of degree 4 and of order 6) with integer coefficients. Substituting (70) into the Killing systems (64) we arrive at the following two commuting two-component flows:

$$\begin{aligned} q_{1,t_3} &= (f_3^5[q_1])_x \\ q_{2,t_3} &= q_1 (f_3^5[q_1])_x - f_3^5[q_1]q_{1,x} + (f_4^5[q_1])_x \end{aligned}$$

and

$$\begin{aligned} q_{1,t_4} &= (f_4^5[q_1])_x \\ q_{2,t_4} &= q_1 (f_4^5[q_1])_x - f_4^5[q_1]q_{1,x} + (f_5^5[q_1])_x \end{aligned}$$

The last vector field can be written more explicitly as

$$\begin{aligned} q_{1,t_4} &= \frac{1}{2^5} (-40ww_xw_{xx} + 35w_x^3 + 8w^2w_{xxx}) w^{-9/2} \\ q_{2,t_4} &= \frac{1}{2^5} (10q_{1,x}ww_x^2 - 8q_{1,x}w^2w_{xx} - 40q_1ww_xw_{xx} + 35q_1w_x^3 + 8q_1w^2w_{xxx}) w^{-9/2} \end{aligned}$$

where  $w = -q_{1,xx}$ . Let us notice that in this case we obtain a hierarchy of systems such that every system is driven by its first equation which is a consecutive equation of Harry Dym hierarchy. One can see that, contrary to the positive case, if  $\alpha > 0$  the obtained systems are always driven by its first  $\sigma$  components that coincide with the corresponding systems from  $\alpha = 0$  hierarchy.

## 8 Conclusions

In this paper we developed a method of unified constructing of Stäckel systems and soliton hierarchies from the same common denominator in the form of separation relations (7). We developed our theory starting from separation relations generated by separation curves of the form

$$H_1\lambda^{\beta_1} + \dots + H_n\lambda^{\beta_n} = \frac{1}{2}\lambda^m\mu^2 + \lambda^k, \quad \beta_i, n \in \mathbf{N}, \quad m, k \in \mathbf{Z}. \quad (71)$$

We performed a detailed, systematic construction of soliton hierarchies for the Benenti class of separation relations, given by the particular form of (71), namely

$$H_1\lambda^{n-1} + H_2\lambda^{n-2} + \dots + H_n = \frac{1}{2}\lambda^m\mu^2 + \lambda^k.$$

The results we obtained are hopefully only a first step of a new research program. The next step of this program would be finding out a way for systematic constructing of other soliton hierarchies from different classes of separation curves (71), when the sequence  $(\beta_1, \dots, \beta_n)$  differs from  $(n-1, \dots, 0)$ . The next - nontrivial - step would be to extend the theory to the case of polynomial separation curves (8) with  $(\alpha_1, \dots, \alpha_n) \neq (0, \dots, 0)$ . We expect by presented procedure to generate not only the majority of known soliton systems but also to construct in a systematic way a vast number of new integrable hierarchies. One should also investigate the possibility of "prolongation" of standard integrable structures of separable systems (such as integrals of motion, bi-Hamiltonian structure) onto the corresponding evolutionary hierarchies of PDE's.

## 9 Appendix A

The involutivity of  $H_1^{(m,k)}$  and  $H_r^{(m,k)}$  (15) leads to the following relations imposed on  $g_{kk}^{(m)}(\lambda)$ ,  $v_r^k(\lambda)$ ,  $V_1^{(k)}(\lambda)$  and  $V_r^{(k)}(\lambda)$  [14]:

$$\frac{\partial v_r^i}{\partial \lambda_i} = 0, \quad i = 1, \dots, n, \quad (\text{A1})$$

$$\frac{\partial}{\partial \lambda_i} \ln g_{kk}^{(m)} = \frac{\frac{\partial v_r^k}{\partial \lambda_i}}{v_r^k - v_r^i}, \quad i \neq k, \quad \text{all } m, r \quad (\text{A2})$$

$$\frac{\partial V_r^{(k)}}{\partial \lambda_i} = v_r^i \frac{\partial V_1^{(k)}}{\partial \lambda_i} \quad \text{for all } i, r \text{ and } k. \quad (\text{A3})$$

We will prove here the relation (20), i.e.

$$\sum_{i=1}^n E_i \left( \mathcal{L}^{(n,m,k)} \right) \lambda_{i,t_r} - \sum_{i=1}^n E_i \left( \mathcal{L}_r^{(n,m,k)} \right) \lambda_{i,x} = 0. \quad (\text{A4})$$

First, let us consider the geodesic case. Due to (18) we have  $\lambda_{i,t_r} = v_r^i \lambda_{i,x}$  where  $v_r^i$  are eigenvalues of  $K_r$  (see (16)). For the geodesic Hamiltonians

$$\mathcal{L}^{(n,m,0)} = \frac{1}{2} \sum_{i=1}^n g_{ii}^{(m)} \lambda_{i,x}^2 \quad \text{and} \quad \mathcal{L}_r^{(n,m,0)} = \frac{1}{2} \sum_{i=1}^n g_{ii}^{(m)} v_r^i \lambda_{i,x}^2$$

so that in this case the left hand side of (A4) attains the form

$$\begin{aligned} & \sum_{i=1}^n E_i \left( \mathcal{L}^{(n,m,0)} \right) \lambda_{i,t_r} - \sum_{i=1}^n E_i \left( \mathcal{L}_r^{(n,m,0)} \right) \lambda_{i,x} \\ &= \left( \frac{1}{2} \sum_{i,k=1}^n \frac{\partial g_{kk}^{(m)}}{\partial \lambda_i} (\lambda_{k,x})^2 v_r^i \lambda_{i,x} - \sum_{i=1}^n \frac{d}{dx} \left( g_{ii}^{(m)} \lambda_{i,x} \right) v_r^i \lambda_{i,x} \right) \\ & - \left( \frac{1}{2} \sum_{i,k=1}^n \frac{\partial g_{kk}^{(m)}}{\partial \lambda_i} (\lambda_{k,x})^2 v_r^k \lambda_{i,x} + \frac{1}{2} \sum_{i,k=1}^n g_{kk}^{(m)} \frac{\partial v_r^k}{\partial \lambda_i} (\lambda_{k,x})^2 \lambda_{i,x} \right. \\ & \left. - \sum_{i=1}^n \frac{d}{dx} \left( g_{ii}^{(m)} \lambda_{i,x} \right) v_r^i \lambda_{i,x} - \sum_{i=1}^n g_{ii}^{(m)} \lambda_{i,x}^2 \frac{dv_r^i}{dx} \right) \\ &= \frac{1}{2} \sum_{i,k=1}^n \frac{\partial g_{kk}^{(m)}}{\partial \lambda_i} (\lambda_{k,x})^2 (v_r^i - v_r^k) \lambda_{i,x} - \left( \frac{1}{2} \sum_{i,k=1}^n g_{kk}^{(m)} \frac{\partial v_r^k}{\partial \lambda_i} (\lambda_{k,x})^2 \lambda_{i,x} - \sum_{i=1}^n g_{ii}^{(m)} \lambda_{i,x}^2 \frac{dv_r^i}{dx} \right) = 0 \end{aligned}$$

since expression in the last parenthesis equals to

$$\begin{aligned} & \frac{1}{2} \sum_{i,k=1}^n g_{kk}^{(m)} \frac{\partial v_r^k}{\partial \lambda_i} (\lambda_{k,x})^2 \lambda_{i,x} - \sum_{i,k=1}^n g_{ii}^{(m)} \lambda_{i,x}^2 \frac{\partial v_r^i}{\partial \lambda_k} \lambda_{k,x} \\ &= -\frac{1}{2} \sum_{i,k=1}^n g_{kk}^{(m)} \frac{\partial v_r^k}{\partial \lambda_i} (\lambda_{k,x})^2 \lambda_{i,x} + \frac{1}{2} \sum_{i,k=1}^n \frac{\partial g_{kk}^{(m)}}{\partial \lambda_i} (\lambda_{k,x})^2 (v_r^i - v_r^k) \lambda_{i,x}, \end{aligned}$$

where the last equality follows from formula (A2) which can be written in equivalent form

$$g_{kk}^{(m)} \frac{\partial v_r^k}{\partial \lambda_i} = \frac{\partial g_{kk}^{(m)}}{\partial \lambda_i} (v_r^k - v_r^i)$$

(that is in fact also valid for  $k = i$ ). Thus, the statement has been proved for geodesic densities. For the potential parts:

$$\sum_{i=1}^n E_i \left( V_1^{(k)} \right) \lambda_{i,t_r} - \sum_{i=1}^n E_i \left( V_r^{(k)} \right) \lambda_{i,x} = \sum_{i=1}^n \left( \frac{\partial V_1^{(k)}}{\partial \lambda_i} v_r^i - \frac{\partial V_r^{(k)}}{\partial \lambda_i} \right) \lambda_{i,x} = 0$$

due to (A3). This concludes the proof.

## 10 Appendix B

We will prove here Theorem 6. The relation (40) is a consequence of (37) and (39) of Lemma 5. Indeed, by (39)

$$E_l(\mathcal{L}^{n,m,k}) = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{(2n-m-i-j)}}{\partial q_l} (q_i)_x (q_j)_x - \frac{\partial V_1^{(k)}}{\partial q_l} - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{(2n-m-i-l)} (q_i)_x \right),$$

and

$$\begin{aligned} E_{l-\alpha}(\mathcal{L}^{n,m+\alpha,k-\alpha}) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{(2n-m-\alpha-i-j)}}{\partial q_{l-\alpha}} (q_i)_x (q_j)_x - \frac{\partial V_1^{(k-\alpha)}}{\partial q_{l-\alpha}} \\ &\quad - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{(2n-m-\alpha-i-l+\alpha)} (q_i)_x \right) \stackrel{\text{lemma 5}}{=} E_l(\mathcal{L}^{n,m,k}). \end{aligned}$$

The relation (41) is just a rewritten form of (??).

Since in what follows we will compare separable potentials with different  $n$ , in the rest of the proof we will use temporary extended notation for potentials in the form  $V_1^{n,(k)}$ . From (28) and (47) it follows that

$$V_1^{n,(n+k)} = V_1^{n+1,(n+1+k)}, \quad k = -n, \dots, n-1. \quad (\text{B1})$$

We prove now (42). Using the relation (37) for  $r = 1$  we obtain

$$\begin{aligned} E_l(\mathcal{L}^{n,0,0}) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(2n-i-j)}}{\partial q_l} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(2n-i-l)} (q_i)_x \right) \\ &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(2n-i-j-l+1)}}{\partial q_l} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(2n-i-l)} (q_i)_x \right) \end{aligned} \quad (\text{B2})$$

and in a similar way we have

$$\begin{aligned} E_{l+1}(\mathcal{L}^{n+1,0,0}) &= \frac{1}{2} \sum_{i,j=1}^{n+1} \frac{\partial V_1^{n+1,(2n+2-i-j-l)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^{n+1} V_1^{n+1,(2n-i-l+1)} (q_i)_x \right) \\ &\stackrel{(*)}{=} \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n+1,(2n-i-j-l+2)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n+1,(2n-i-l+1)} (q_i)_x \right) \\ &\stackrel{(\text{B1})}{=} \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(2n-i-j-l+1)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(2n-i-l)} (q_i)_x \right). \end{aligned}$$

The equality (\*) is due to the fact that  $V_1^{n+1,(2n-i-l+1)} = 0$  for  $i = n+1$  and similarly  $V_1^{n+1,(2n-i-j-l+2)}$  does not depend on  $q_1$  for  $i = n+1$  or  $j = n+1$  (this follows from (42) and (57)). Thus

$$E_l(\mathcal{L}^{n,0,0}) = E_{l+1}(\mathcal{L}^{n+1,0,0}), \quad l = 1, \dots, n. \quad (\text{B3})$$

Moreover, from (30) it follows that

$$V_1^{n,(2n+\sigma)} = -q_1 V_1^{n,(2n+\sigma-1)} - \dots - q_n V_1^{n,(n+\sigma)},$$

hence, for  $l > \sigma$

$$\begin{aligned} \frac{\partial V_1^{n,(2n+\sigma)}}{\partial q_l} &= -q_1 \frac{\partial V_1^{n,(2n+\sigma-1)}}{\partial q_l} - \dots - q_n \frac{\partial V_1^{n,(n+\sigma)}}{\partial q_l} - V_1^{n,(2n+\sigma-l)} \\ &= -q_1 \frac{\partial V_1^{n,(2n+\sigma-l)}}{\partial q_1} - \dots - q_n \frac{\partial V_1^{n,(n+\sigma-l+1)}}{\partial q_1} - V_1^{n,(2n+\sigma-l)}. \end{aligned}$$

On the other hand we have

$$V_1^{n+1,(2n+\sigma+2)} = -q_1 V_1^{n+1,(2n+\sigma+1)} - \dots - q_n V_1^{n+1,(n+\sigma+2)} - q_{n+1} V_1^{n+1,(n+\sigma+1)},$$

hence, for  $l > \sigma$  and according to (B1) and (47)

$$\begin{aligned} \frac{\partial V_1^{n+1,(2n+\sigma+2)}}{\partial q_{l+1}} &= -q_1 \frac{\partial V_1^{n+1,(2n+\sigma+1)}}{\partial q_{l+1}} - \dots - q_n \frac{\partial V_1^{n+1,(n+\sigma+2)}}{\partial q_{l+1}} - V_1^{n+1,(2n+\sigma-l+1)} \\ &= -q_1 \frac{\partial V_1^{n+1,(2n+\sigma+1-l)}}{\partial q_1} - \dots - q_n \frac{\partial V_1^{n+1,(n+\sigma+2-l)}}{\partial q_1} - V_1^{n+1,(2n+\sigma-l+1)} \\ &= -q_1 \frac{\partial V_1^{n,(2n+\sigma-l)}}{\partial q_1} - \dots - q_n \frac{\partial V_1^{n,(n+\sigma+1-l)}}{\partial q_1} - V_1^{n,(2n+\sigma-l)} \\ &= \frac{\partial V_1^{n,(2n+\sigma)}}{\partial q_l}, \end{aligned} \tag{B4}$$

and from (37) it follows that

$$\frac{\partial V_1^{n,(n+s)}}{\partial q_l} = \frac{\partial V_1^{n+1,(n+s+2)}}{\partial q_{l+1}}, \quad 0 \leq s - l + 1 \leq n.$$

So, from (B3) and (B4) for  $\sigma < l \leq n$  equation (42) is fulfilled.

Now, we pass to the proof of relations (43). First, we have

$$\begin{aligned} E_l(\mathcal{L}^{n,n-\sigma,0}) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(n+\sigma-i-j)}}{\partial q_l} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(n+\sigma-i-l)} (q_i)_x \right) \\ &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(n+\sigma-i-j-l+1)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(n+\sigma-i-l)} (q_i)_x \right). \end{aligned} \tag{B5}$$

On the other hand

$$E_l(\mathcal{L}^{n+1,n+1-\sigma,0}) = \frac{1}{2} \sum_{i,j=1}^{n+1} \frac{\partial V_1^{n+1,(n+\sigma-i-j-l+2)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^{n+1} V_1^{n+1,(n+\sigma-i-l+1)} (q_i)_x \right).$$

By (42) and (57), for  $l \leq \sigma$  the last term in both sums does not contribute. Moreover, according to (B1) and the fact that  $V_1^{n,(2n)} - q_{n+1} = V_1^{n+1,(2n+1)}$ , we have

$$\frac{\partial V_1^{n,(n+k)}}{\partial q_1} = \frac{\partial V_1^{n+1,(n+k+1)}}{\partial q_1}, \quad k = -n, \dots, n,$$

hence

$$\begin{aligned} E_l(\mathcal{L}^{n+1,n+1-\sigma,0}) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(n+\sigma-i-j-l+1)}}{\partial q_1} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(n+\sigma-i-l)} (q_i)_x \right) \\ &= E_l(\mathcal{L}^{n,n-\sigma,0}). \end{aligned}$$

Finally, we prove the relation (44). From the negative recursion (27), we have

$$V_1^{n,(-k)}(q_{n-k+1}, \dots, q_n) \Big|_{q_i \rightarrow q_{i+1}, i=n-k+1, \dots, n} = V_1^{n+1,(-k)}(q_{n-k+2}, \dots, q_{n+1}), \quad k = 1, \dots, n. \tag{B6}$$

Then

$$\begin{aligned} E_{\sigma+l}(\mathcal{L}^{n,n-\sigma,0}) &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(n-i-j+1-\sigma)}}{\partial q_{l+\sigma}} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(n-i-l)} (q_i)_x \right) \\ &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial V_1^{n,(2n-i-j-l+1)}}{\partial q_n} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^n V_1^{n,(n-i-l)} (q_i)_x \right) \end{aligned} \quad (\text{B7})$$

and

$$\begin{aligned} E_{\sigma+l}(\mathcal{L}^{n+1,n+1-\sigma,0}) &= \frac{1}{2} \sum_{i,j=1}^{n+1} \frac{\partial V_1^{n+1,(n-i-j+2-\sigma)}}{\partial q_{l+\sigma}} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^{n+1} V_1^{n+1,(n-i-l+1)} (q_i)_x \right) \\ &\quad \frac{1}{2} \sum_{i,j=1}^{n+1} \frac{\partial V_1^{n+1,(2n-i-j-l+3)}}{\partial q_{n+1}} (q_i)_x (q_j)_x - \frac{d}{dx} \left( \sum_{i=1}^{n+1} V_1^{n+1,(n-i-l+1)} (q_i)_x \right) \\ &\quad \frac{1}{2} \sum_{i,j=0}^n \frac{\partial V_1^{n+1,(2n-i-j-l+1)}}{\partial q_{n+1}} (q_{i+1})_x (q_{j+1})_x - \frac{d}{dx} \left( \sum_{i=0}^n V_1^{n+1,(n-i-l)} (q_{i+1})_x \right). \end{aligned}$$

As for  $i, j = 0$  there is no contribution to the sum, so according to (B6) we have

$$E_{\sigma+l}(\mathcal{L}^{n,n-\sigma,0})|_{q_j \rightarrow q_{j+1}} = E_{\sigma+l}(\mathcal{L}^{n+1,n+1-\sigma,0}), \quad l = 1, \dots, n - \sigma, \quad j = 1, \dots, n. \quad (\text{B8})$$

From (37) and (B6) we have

$$\left. \frac{\partial V_1^{n,(-n)}}{\partial q_i} \right|_{q_j \rightarrow q_{j+1}, \quad j=1, \dots, n} = \frac{\partial V_1^{n+1,(-n)}}{\partial q_{i+1}} = \frac{\partial V_1^{n+1,(-n-1)}}{\partial q_i}, \quad i = 1, \dots, n. \quad (\text{B9})$$

that together with (B8) proves the relation (44).

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